

GEOMETRICAL VARIATIONS OF A STATE-CONSTRAINED FUNCTIONAL ON STAR-SHAPED DOMAINS

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Abstract. The aim of this paper is to compute the shape derivative of a volume cost functional subject to a Dirichlet boundary value state constraint problem using the shape derivative formula introduced in (Boulkhemair et al., 2020), which concerns star-shaped domains. This allows us to express the shape derivative by means of support functions of convex domains. Then we give a simple algorithm for the numerical resolution of this problem.

Keywords: shape optimization, shape derivative, star-shaped domains, convex domains, support function.

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1 Introduction

Shape optimization is a part of the field of optimal control theory. The main objective in shape optimization problems is to deform the outer boundary of an object in order to minimize or maximize a cost function, while satisfying given constraints. Historically, the shape optimization methods have been used in cutting edge technologies mainly in advanced areas such as aerodynamics. They have recently been extended to other engineering areas where the shape greatly influences the performances, for example, in hydrodynamics, elasticity, geophysics or mechanical models (Allaire, 2003; Boulkhemair et al., 2013; Henrot & Pierre, 2006; Pironneau, 1984). Indeed, the shape optimization is now commonly used for solving problems that are related to a variety of phenomena in different scientist sectors, in order to improve the productivity, reduce the cost and maximize the profit.

In many cases, the shape optimization problem is reduced to find an optimal shape by minimizing a certain cost functional, subject to given constraints, which often depends on the solution of a given partial differential equation defined on the variable domain. Generally, we try to solve and analyze problems of the following kind: find a solution Ω^* of

$$\Omega^* \in \mathcal{O}, \quad \mathcal{J}(\Omega^*) = \inf_{\Omega \in \mathcal{O}} \mathcal{J}(\Omega),$$

where \mathcal{O} is a class of subsets in \mathbb{R}^n and \mathcal{J} is a functional defined on \mathcal{O} with values in \mathbb{R} . The elements of \mathcal{O} are called admissible shapes or domains and \mathcal{J} is called a shape or cost functional.

At the beginning of any optimization process, there is a modeling question. One has to choose a mathematical model to represent the data to be optimized. There are two main ingredients in a mathematical model for shape optimization : at first the way to represent a shape, and secondly

the way to perform a sensitivity analysis. In this work we are interested in a new method in sensitivity analysis. Indeed, the numerical investigation of shape optimization problems is based on the study of the first variation of the cost functional, and in particular on the computation of its gradient or what one call in the literature the shape derivative. This notion was first introduced by Hadamard in his famous memory (Hadamard, 1907). We recall the two usual frameworks for computing shape derivatives with the Hadamard method of variation of domains using vector fields : the displacement field method and the speed method. A shape is considered as a bounded open set of \mathbb{R}^n , so if $\theta \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ a variation of the reference shape Ω is defined by

$$\Omega_\theta = (\text{Id}_{\mathbb{R}^n} + \theta)(\Omega) = \{x + \theta(x) \mid x \in \Omega\}.$$

Then differentiating with respect to θ defines the shape derivative with respect to the displacement field method (Allaire, 2003; C ea, 1964; Murat & Simon, 1974, 1976). For the speed method : if $V \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n)$ is a vector field, we can consider the solution to the following equation

$$\Phi_V(0, x) = x \quad \text{and} \quad \frac{d\Phi_V(t, x)}{dt} = V(t, \Phi_V(t, x)), \quad x \in \Omega. \quad (1)$$

This defines a time-dependent domain

$$\Omega_t = \Phi_V(t, \Omega) = \{\Phi_V(t, x) \mid x \in \Omega\}.$$

Then differentiating with respect to the time parameter leads to another notion of shape derivative (Henrot & Pierre, 2006; Delfour & Zol esio, 2011; Sokolowski & Zolesio, 1992).

But these techniques themselves present some difficulties from both theoretical and numerical point of view. For example, when one wants to connect the set of admissible domains with vector fields, one has to suppose high smoothness conditions on the initial data in order to differentiate functions depending on the domain. The main objective in this paper is to develop a new method for the shape differentiability (Niftiyev & Gasimov, 2004; Boulkhemair, 2003; Boulkhemair & Chakib, 2014, 2015; Boulkhemair et al., 2020) for a shape optimization problem of a volume cost functional subject to a boundary value problem. Then we establish the expression for its shape derivative via support functions, using the formula of shape derivative with respect to star-shaped domains proposed in (Boulkhemair et al., 2020). This formula was in fact introduced the first time by A. A. Niftiyev and Y. Gasimov (Niftiyev & Gasimov, 2004) for convex domains and studied and developed by A. Boulkhemair, A. Chakib and A. Sadik (Boulkhemair, 2003; Boulkhemair & Chakib, 2014, 2015; Boulkhemair et al., 2020). In order to be more precise, let Ω_0 be a bounded star-shaped domains of class C^2 , Ω be a bounded convex domain of class C^2 and a family of functions $(f_\epsilon)_\epsilon \subset L^1_{loc}(\mathbb{R}^n)$ with f_0 in the Sobolev space $W^{1,1}_{loc}(\mathbb{R}^n)$ and let f be a function such that

$$\frac{f_\epsilon - f_0}{\epsilon} \rightarrow f \text{ in } L^1_{loc}(\mathbb{R}^n) \text{ as } \epsilon \rightarrow 0^+.$$

Then,

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \left(\int_{\Omega_0 + \epsilon\Omega} f_\epsilon dx - \int_{\Omega_0} f_0 dx \right)$$

exists and is equal to

$$\int_{\Omega_0} f(x) dx + \int_{\partial\Omega_0} f_0(x) P_\Omega(\nu_0(x)) d\sigma(x), \quad (2)$$

where $\nu_0(x)$ denotes the outward unit normal vector to $\partial\Omega_0$ at x , and P_Ω is the support function of the convex domain Ω .

As said above, our interest in a such formula came first from a numerical study undertaken in (Boulkhemair et al., 2021, 2020). In fact, we believe that the use of support functions is more advantageous than that using vectors fields. We refer, for example, to (Allaire, 2003), for

explanations about the difficulties that arise when implementing numerically the minimization of domain integral functionals, via gradient method type's, using the usual expression of the shape derivative by vector fields. In fact, when using vector fields, we have to extend the vector field (obtained only on the boundary) to all the domain or to re-mesh at each iteration of the process, and both approaches are expensive. While for this proposed approach involving support functions, we get not only a set of boundary points but also a support function, at each iteration. Then by taking its subdifferential at the origin, we get the next domain.

The outline of the paper is as follows. In the second section, we present the considered shape optimization problem. In the third section, we give some preliminary results on the shape derivative formulas using Minkowski deformation for a volume cost functional. In the fourth section, we give the main result of this work which is the computation of the shape derivative of the cost functional on the considered family of admissible domains and establish the expression for its shape derivative by means of support functions. In the last section, we describe in more details the main ingredients of the proposed process of optimization and we propose an algorithm for the approximation of the problem, based on a gradient method.

2 Statement of the shape optimization problem

We are concerned with the following typical shape optimization problem :

$$\min_{\Omega \in \mathcal{U}} J(\Omega, u_\Omega) \quad (3)$$

where

$$J(\Omega, u_\Omega) := \int_{\Omega} j(x, u_\Omega, \nabla u_\Omega) dx \quad (4)$$

and u_Ω satisfies

$$Au_\Omega = f \quad \text{in } \Omega, \quad (5)$$

$$A_b u_\Omega = g \quad \text{on } \Gamma = \partial\Omega, \quad (6)$$

where f and g are given functions, A and A_b are given operators, \mathcal{U} denotes the set of admissible domains and j is a function that do not depend on the shape Ω .

In the sequel, we will propose a numerical method for the approximation of this problem based on a gradient method. This requires to study the first variation or differentiability of the cost functional with respect to an appropriate family of domains \mathcal{U} . For this, we will use the shape derivative formula proposed in (Boulkhemair et al., 2020). So let us define the set of admissible domains. Let D be a fixed smooth and bounded open subset of \mathbb{R}^n . The set of admissible domains \mathcal{U} is the set of bounded open subset of \mathbb{R}^n which are of class C^2 and star-shaped with respect to some ball of radius $r > 0$.

In this work, we will only consider the following types of functionals. The first one is:

$$\Omega \in \mathcal{U} \mapsto \int_{\Omega} |u_\Omega - \varphi_0|^2 dx, \quad (7)$$

while the second one involves the gradient operator:

$$\Omega \in \mathcal{U} \mapsto \int_{\Omega} \|\nabla u_\Omega - \nabla \varphi_1\|^2 dx. \quad (8)$$

In fact, it is equivalent to study the functional

$$\mathcal{J}(\Omega, u_\Omega) = \int_{\Omega} j(u_\Omega, \nabla u_\Omega) dx, \quad (9)$$

$$\text{with } j(u_\Omega, \nabla u_\Omega) = \alpha |u_\Omega - \varphi_0|^2 + \beta \|\nabla u_\Omega - \nabla \varphi_1\|^2, \quad (10)$$

where α and β are fixed real numbers. Here, $\|\cdot\|$ denotes the euclidian norm in \mathbb{R}^n and u_Ω is the solution of the state equation associated to the operators $A = -\Delta$ and $A_b = 1$. The given functions f, g, φ_0 and φ_1 satisfy appropriate regularity assumptions allowing the existence of the shape derivative of J with respect to Ω .

3 Shape sensitivity analysis using Minkowski deformation

In order to compute the shape derivative of the cost functional for the shape optimization problem (3), we recall first the result on the shape derivative formulas given in (Boulkhemair, 2003; Boulkhemair & Chakib, 2014; Boulkhemair et al., 2020) for the class of star-shaped domains \mathcal{U} .

3.1 Preliminary results

Consider a real-valued shape function

$$J : \Omega \in \mathcal{U} \mapsto J(\Omega) \in \mathbb{R}$$

defined on a family \mathcal{U} of subsets of \mathbb{R}^n .

Let \mathcal{O} be the set of convex domains of class C^2 and \mathcal{K} denote the set of all convex domains.

Since we are interested in the derivative with respect to the shape, let us first define the technique adopted for the deformation of domains based on Minkowski sum and then define the associated shape derivative.

Definition 1. *Let $\Omega \in \mathcal{U}$ and $\Theta \in \mathcal{O}$. The deformed domain denoted by Ω_ϵ is given by the sum of Minkowski as follows :*

$$\Omega_\epsilon = \Omega + \epsilon\Theta := \{x + \epsilon y \mid x \in \Omega, y \in \Theta\}, \quad \epsilon \in [0, 1].$$

A shape functional J is called shape differentiable at Ω in the direction of Θ , if the eulerian derivative

$$\delta J(\Omega)[\Theta] := \lim_{\epsilon \rightarrow 0^+} \frac{J(\Omega_\epsilon) - J(\Omega)}{\epsilon}, \quad \Omega_\epsilon = \Omega + \epsilon\Theta$$

exists for all $\Theta \in \mathcal{O}$. Then the expression $\delta J(\Omega)[\Theta]$ is called the shape derivative of J at Ω in the direction of Θ .

To our knowledge, this kind of deformation was first used in the field of shape optimization by A. A. Niftiyev and Y. Gasimov (Niftiyev & Gasimov, 2004). More precisely, they proposed the deformation

$$(1 - \epsilon)\Omega + \epsilon\Theta, \quad \text{for } \Omega, \Theta \in \mathcal{O} \text{ and } \epsilon \in [0, 1],$$

to express the shape derivative of a volume cost functional, under appropriate regularity assumptions, by means of support functions of convex domains. Then, inspired by the Brunn-Minkowski theory (see, for example, R. Schneider, (Schneider, 2014)), A. Boulkhemair and A. Chakib (Boulkhemair, 2003; Boulkhemair & Chakib, 2014) proposed to compute the shape derivative by considering the Minkowski deformation

$$\Omega + \epsilon\Theta, \quad \text{for } \Omega \in \mathcal{U}, \Theta \in \mathcal{O} \text{ and } \epsilon \in [0, 1].$$

In the sequel, we will opt for the last technique of deformation.

In this context, let us recall the shape derivative formula for a volume integral shape functional J of type

$$\Omega \in \mathcal{U} \mapsto J(\Omega) = \int_{\Omega} g(x)dx,$$

where g is in the Sobolev space $W^{1,1}(D)$.

Recall first that the support function P_Θ of a bounded convex domain Θ is given by a continuous, convex and positively homogeneous function:

$$P_\Theta(x) = \sup_{y \in \Theta} \langle x, y \rangle = \sup_{y \in \bar{\Theta}} \langle x, y \rangle, \quad x \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product of x and y in \mathbb{R}^n . Conversely, for any continuous, convex, positively homogeneous function $P(x)$ there exists a unique convex bounded set Ω , such that $P(x)$ is its support function, i.e. $P(x) = P_\Omega(x)$, where Ω is obtained as a sub-differential of the function P at the origin :

$$\bar{\Omega} = \partial P(0) := \{\xi \in \mathbb{R}^n \mid \langle \xi, y \rangle \leq P(y), \forall y \in \mathbb{R}^n\}.$$

Now, according to (Boukhemair et al., 2020), we have

Theorem 1. *Consider the set \mathcal{U} of domains which are star-shaped with respect to some ball and are contained in D . Let $\Omega \in \mathcal{U}$ and $\Theta \in \mathcal{O}$. Then, the shape derivative of J at $\Omega \in \mathcal{U}$ in the direction Θ exists and is given by*

$$\lim_{\epsilon \rightarrow 0^+} \frac{J(\Omega_\epsilon) - J(\Omega)}{\epsilon} = \int_{\partial\Omega} g(x) P_\Theta(\nu(x)) d\sigma(x),$$

where $\Omega_\epsilon = \Omega + \epsilon\Theta$ and ν denotes the exterior unit normal vector to Ω .

In the situation where the function g depends on domains, one can show the following more general result.

Proposition 1. *Let $(g_\epsilon)_{\epsilon \in]0,1[} \subset L^1(D)$ be a family of functions and let $g_0 \in W^{1,1}(D)$ and g be a function such that*

$$g = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} (g_\epsilon - g_0) \text{ in } L^1(D).$$

Consider the function

$$\epsilon \in]0, 1[\mapsto I(\epsilon) = \int_{\Omega_\epsilon} g_\epsilon(x) dx \in \mathbb{R}.$$

Then we have

$$\lim_{\epsilon \rightarrow 0^+} \frac{I(\epsilon) - I(0)}{\epsilon} = \int_{\Omega} g(x) dx + \int_{\Gamma} g_0(x) P_\Theta(\nu(x)) d\sigma(x). \quad (11)$$

where ν denotes the outward unit normal vector to $\Gamma = \partial\Omega$.

Proof. We can write

$$\begin{aligned} \frac{I(\epsilon) - I(0)}{\epsilon} &= \frac{1}{\epsilon} \left(\int_{\Omega_\epsilon} g_\epsilon(x) dx - \int_{\Omega_\epsilon} g_0(x) dx \right) + \frac{1}{\epsilon} \left(\int_{\Omega_\epsilon} g_0(x) dx - \int_{\Omega_0} g_0(x) dx \right) \\ &= \int_D \chi_{\Omega_\epsilon} \left(\frac{1}{\epsilon} (g_\epsilon - g_0)(x) - f(x) \right) dx + \int_D \chi_{\Omega_\epsilon}(x) g(x) dx + \frac{1}{\epsilon} \left(\int_{\Omega_\epsilon} g_0(x) dx - \int_{\Omega_0} g_0(x) dx \right). \end{aligned}$$

First, we have

$$\left| \int_D \chi_{\Omega_\epsilon} \left(\frac{1}{\epsilon} (g_\epsilon - g_0)(x) - g(x) \right) dx \right| \leq \int_D \left| \frac{1}{\epsilon} (g_\epsilon - g_0)(x) - g(x) \right| dx \xrightarrow{\epsilon \rightarrow 0^+} 0.$$

On the other hand, since we have that $\chi_{\Omega_\epsilon} = \chi_{\Omega + \epsilon\Theta}$ and that the characteristic functions χ_{Ω_ϵ} converge almost everywhere to the characteristic function χ_Ω , then from the Lebesgue convergence theorem in $L^1(D)$ and by the use of Theorem 1, it follows that

$$\lim_{\epsilon \rightarrow 0^+} \frac{I(\epsilon) - I(0)}{\epsilon} = \int_D \chi_\Omega g(x) dx + \int_\Gamma g_0(x) P_\Theta(\nu(x)) d\sigma(x).$$

Consequently, we get

$$\lim_{\epsilon \rightarrow 0^+} \frac{I(\epsilon) - I(0)}{\epsilon} = \int_{\Omega} g(x) dx + \int_{\Gamma} g_0(x) P_{\Theta}(\nu(x)) d\sigma(x)$$

□

The following result concerns the situation where g is written as a product of two functions depending on the domains.

Proposition 2. *Let $(f_{\epsilon})_{\epsilon \in]0,1[}$ and $(k_{\epsilon})_{\epsilon \in]0,1[}$ be two families of functions in $L^2(D)$ and let $f_0 \in H^1(D)$, $k_0 \in H^1(D)$ and f, k be functions such that*

$$f = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} (f_{\epsilon} - f_0) \text{ in } L^2(D) \quad \text{and} \quad k = \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} (k_{\epsilon} - k_0) \text{ in } L^2(D).$$

Consider the function

$$\epsilon \in]0, 1[\mapsto F(\epsilon) = \int_{\Omega_{\epsilon}} f_{\epsilon}(x) k_{\epsilon}(x) dx \in \mathbb{R}.$$

Then, we have

$$\lim_{\epsilon \rightarrow 0^+} \frac{F(\epsilon) - F(0)}{\epsilon} = \int_{\Omega} (k f_0 + f k_0)(x) dx + \int_{\Gamma} (f_0 k_0)(x) P_{\Theta}(\nu(x)) d\sigma(x), \quad (12)$$

where ν denotes the exterior unit normal vector to Ω .

Proof. We can write

$$\frac{1}{\epsilon} (f_{\epsilon} k_{\epsilon} - f_0 k_0) - f_0 k - f k_0 = f (k_{\epsilon} - k_0) + k_{\epsilon} \left(\frac{1}{\epsilon} (f_{\epsilon} - f_0) - f \right) + f_0 \left(\frac{1}{\epsilon} (k_{\epsilon} - k_0) - k \right).$$

Using Cauchy-Schwarz inequality, we get

$$\begin{aligned} \left\| \frac{1}{\epsilon} (f_{\epsilon} k_{\epsilon} - f_0 k_0) - f_0 k - f k_0 \right\|_{L^1(D)} &\leq \|f\|_{L^2(D)} \|k_{\epsilon} - k_0\|_{L^2(D)} + \|k_{\epsilon}\|_{L^2(D)} \left\| \frac{f_{\epsilon} - f_0}{\epsilon} - f \right\|_{L^2(D)} \\ &\quad + \|f_0\|_{L^2(D)} \left\| \frac{k_{\epsilon} - k_0}{\epsilon} - k \right\|_{L^2(D)}. \end{aligned}$$

It follows from the assumptions that $\|k_{\epsilon} - k_0\|_{L^2(D)}$ converge to 0 as $\epsilon \rightarrow 0$. Therefore, there exists $M > 0$ such that $\|k_{\epsilon}\|_{L^2(D)} \leq M$ for small enough ϵ . Consequently,

$$\lim_{\epsilon \rightarrow 0^+} \left\| \frac{1}{\epsilon} (f_{\epsilon} k_{\epsilon} - f_0 k_0) - f_0 k - f k_0 \right\|_{L^1(D)} = 0.$$

So, applying Proposition 1 to the functional

$$\epsilon \in]0, 1[\mapsto F(\epsilon) = \int_{\Omega_{\epsilon}} f_{\epsilon}(x) k_{\epsilon}(x) dx \in \mathbb{R},$$

yields

$$\lim_{\epsilon \rightarrow 0^+} \frac{F(\epsilon) - F(0)}{\epsilon} = \int_{\Omega} (f_0 k + f k_0)(x) dx + \int_{\Gamma} f_0(x) k_0(x) P_{\Theta}(\nu(x)) d\sigma(x). \quad (13)$$

□

4 Shape derivative under a state constraint problem

In this section, we prove and state the main result of this paper. Recall that we are interested in computing the shape derivative of the shape cost functional

$$\mathcal{J}(\Omega, u_\Omega) = \alpha \int_{\Omega} |u_\Omega - \varphi_0|^2 dx + \beta \int_{\Omega} \|\nabla u_\Omega - \nabla \varphi_1\|^2 dx, \quad \Omega \in \mathcal{U}, \quad (14)$$

where α and β are fixed real numbers and the family \mathcal{U} of admissible domains is the set of domains which are star-shaped with respect to some ball and are contained in D and u_Ω is the solution of the following state problem on Ω

$$\begin{cases} -\Delta u_\Omega = f & \text{in } \Omega, \\ u_\Omega = g & \text{on } \Gamma = \partial\Omega. \end{cases} \quad (15)$$

In the sequel, for simplicity, we assume that $g = 0$ and the given functions f , φ_0 and φ_1 satisfy the following regularity assumptions

$$\mathbf{(H)} \quad f \in H^1(D), \quad \varphi_0 \in H_{loc}^1(\mathbb{R}^n) \quad \text{and} \quad \varphi_1 \in H_{loc}^2(\mathbb{R}^n).$$

Note that these regularity assumptions ensure the well posedness of the state problem and allow at the same time the existence of the shape derivative.

Now, let $\Omega \in \mathcal{U}$, $\epsilon \in [0, 1]$ and let $\Theta \in \mathcal{O}$, that is, Ω is a convex domain of class C^2 contained in D . We denote by $\Omega_\epsilon = \Omega + \epsilon\Theta$ the Minkowski deformation domain of Ω by Θ . Let u_{Ω_ϵ} be the solution of the state problem on Ω_ϵ

$$\begin{cases} -\Delta u_{\Omega_\epsilon} = f & \text{in } \Omega_\epsilon, \\ u_{\Omega_\epsilon} = 0 & \text{on } \Gamma_\epsilon = \partial\Omega_\epsilon. \end{cases} \quad (16)$$

According to (Boukhemair et al., 2020), recall that, if Θ is a strongly convex domain, Ω_ϵ can be considered as a deformation of the domain Ω by the vector field $V(x) = a(x)$ such that

$$\langle a(x), \nu_\Gamma(x) \rangle = P_\Theta(\nu_\Gamma(x)),$$

where ν_Γ denotes the exterior unit normal vector to Ω . In the sequel, we will need the following result (see for example, (Henrot & Pierre, 2006; Delfour & Zolésio, 2011; Sokolowski & Zolesio, 1992))

Theorem 2. *Assume that the assumptions $\mathbf{(H)}$ hold and that Θ is a strongly convex domain. Let $u = u_\Omega$ be the unique solution of (15) on Ω . Then, for $\epsilon \in [0, 1]$, the unique solution $u_\epsilon = u_{\Omega_\epsilon}$ of (15) on Ω_ϵ satisfies*

$$\tilde{u}_\epsilon = \tilde{u} + \epsilon u' + \epsilon U_\epsilon \quad \text{where } U_\epsilon \rightarrow 0 \quad \text{in } H^1(D)$$

where \tilde{u}_ϵ and \tilde{u} designate respectively extensions on D of u_ϵ and u . Moreover, the functions $j'_0 = 2u'(\tilde{u} - \varphi_0)$ and $j'_1 = 2\langle \nabla u', \nabla \tilde{u} - \nabla \varphi_1 \rangle$ satisfy

$$\frac{1}{\epsilon} [|\tilde{u}_\epsilon - \varphi_0|^2 - |\tilde{u} - \varphi_0|^2] - j'_0 \rightarrow 0 \quad \text{in } L^1(D), \quad \epsilon \rightarrow 0$$

and

$$\frac{1}{\epsilon} [\|\nabla \tilde{u}_\epsilon - \nabla \varphi_1\|^2 - \|\nabla \tilde{u} - \nabla \varphi_1\|^2] - j'_1 \rightarrow 0 \quad \text{in } L^1(D), \quad \epsilon \rightarrow 0.$$

In particular, we get the existence of the shape derivative of \mathcal{J} .

4.1 Shape derivative of the cost functional

Let $\epsilon \in [0, 1]$ and let u_Ω and u_{Ω_ϵ} be respectively the solution of (15) and (16). Consider their extensions \tilde{u}_Ω and $\tilde{u}_{\Omega_\epsilon}$ on D that we denote, for simplicity, again by u_Ω and u_{Ω_ϵ} .

Let's denote by j_0 and j_1 the following functions:

$$\begin{aligned} j_0 : D \times \mathbb{R} &\longrightarrow \mathbb{R} & j_1 : D \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto j_0(x, y) = |y - \varphi_0(x)|^2 & (x, y) &\longmapsto j_1(x, y) = \|y - \nabla \varphi_1(x)\|^2, \end{aligned}$$

so that

$$\mathcal{J}(\Omega, u_\Omega) = \alpha \int_{\Omega} j_0(x, u_\Omega(x)) dx + \beta \int_{\Omega} j_1(x, \nabla u_\Omega(x)) dx.$$

Now, let us define $\Delta \mathcal{J}(\Omega, u_\Omega) = \mathcal{J}(\Omega_\epsilon, u_{\Omega_\epsilon}) - \mathcal{J}(\Omega, u_\Omega)$. We have

$$\Delta \mathcal{J}(\Omega, u_\Omega) = \alpha \int_{\Omega_\epsilon} j_0(\cdot, u_{\Omega_\epsilon}) dx + \beta \int_{\Omega_\epsilon} j_1(\cdot, \nabla u_{\Omega_\epsilon}) dx - \alpha \int_{\Omega} j_0(\cdot, u_\Omega) dx - \beta \int_{\Omega} j_1(\cdot, \nabla u_\Omega) dx.$$

Define then the functions J_1, J_2, J_3 and J_4 by

$$\begin{aligned} J_1(\epsilon) &= \int_{\Omega_\epsilon} j_0(\cdot, u_{\Omega_\epsilon}) dx - \int_{\Omega} j_0(\cdot, u_\Omega) dx, & J_2(\epsilon) &= \int_{\Omega} (j_0(\cdot, u_{\Omega_\epsilon}) - j_0(\cdot, u_\Omega)) dx \\ J_3(\epsilon) &= \int_{\Omega_\epsilon} j_1(\cdot, \nabla u_{\Omega_\epsilon}) dx - \int_{\Omega} j_1(\cdot, \nabla u_\Omega) dx, & J_4(\epsilon) &= \int_{\Omega} (j_1(\cdot, \nabla u_{\Omega_\epsilon}) - j_1(\cdot, \nabla u_\Omega)) dx, \end{aligned}$$

so that

$$\Delta \mathcal{J}(\Omega, u_\Omega) = \alpha(J_1(\epsilon) + J_2(\epsilon)) + \beta(J_3(\epsilon) + J_4(\epsilon)).$$

Let us first compute the shape derivative of the functions J_1 and J_3 . Setting $\delta u = u_{\Omega_\epsilon} - u_\Omega$, we have

$$\begin{aligned} j_0(\cdot, u_{\Omega_\epsilon}) - j_0(\cdot, u_\Omega) &= |u_{\Omega_\epsilon} - \varphi_0|^2 - |u_\Omega - \varphi_0|^2 \\ &= |\delta u|^2 + 2(u_\Omega - \varphi_0)\delta u. \end{aligned} \tag{17}$$

We note that since $f \in H^1(D)$ and the domains D, Ω and Θ are smooth enough, we have $u_\Omega \in H^2(\Omega)$ and $u_{\Omega_\epsilon} \in H^2(\Omega_\epsilon)$. On the other hand, according to Theorem 2, there exists $u' \in H^1(D)$ such that

$$\left\| \frac{u_{\Omega_\epsilon} - u_\Omega}{\epsilon} - u' \right\|_{H^1(D)} \xrightarrow{\epsilon \rightarrow 0^+} 0. \tag{18}$$

Now, we have

$$\begin{aligned} \left\| \frac{j_0(\cdot, u_{\Omega_\epsilon}) - j_0(\cdot, u_\Omega)}{\epsilon} - 2(u_\Omega - \varphi_0)u' \right\|_{L^1(D)} &\leq \frac{1}{\epsilon} \|\delta u\|_{L^2(D)}^2 + 2 \int_D |(u_\Omega - \varphi_0) \left(\frac{\delta u}{\epsilon} - u' \right)| dx \\ &\leq \frac{1}{\epsilon} \|\delta u\|_{L^2(D)}^2 + 2\|u_\Omega - \varphi_0\|_{L^2(D)}^2 \left\| \frac{\delta u}{\epsilon} - u' \right\|_{L^2(D)}^2 \end{aligned}$$

and, by using (18), we have

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \|\delta u\|_{L^2(D)}^2 = 0 \quad \text{and} \quad \lim_{\epsilon \rightarrow 0^+} 2\|u_\Omega - \varphi_0\|_{L^2(D)}^2 \left\| \frac{\delta u}{\epsilon} - u' \right\|_{L^2(D)}^2 = 0.$$

Thus,

$$\lim_{\epsilon \rightarrow 0^+} \left\| \frac{j_0(\cdot, u_{\Omega_\epsilon}) - j_0(\cdot, u_\Omega)}{\epsilon} - 2(u_\Omega - \varphi_0)u' \right\|_{L^1(D)} = 0,$$

that is, the function $\epsilon \mapsto j_0(\cdot, u_{\Omega_\epsilon}) \in L^1(D)$ is differentiable at 0^+ .

Setting $j'_0 = 2(u_\Omega - \varphi_0)u'$ and applying Proposition 1 to the function $J_1(\epsilon)$, we obtain

$$\begin{aligned} \frac{d}{d\epsilon} (J_1(\epsilon)) \Big|_{\epsilon=0^+} &= \int_{\Omega} \frac{d}{d\epsilon} j_0(\cdot, u_{\Omega_\epsilon}) \Big|_{\epsilon=0^+} dx + \int_{\Gamma} j_0(\cdot, u_\Omega) P_\Theta(\nu) d\sigma - \int_{\Omega} \frac{d}{d\epsilon} j_0(\cdot, u_{\Omega_\epsilon}) \Big|_{\epsilon=0^+} dx \\ &= \int_{\Omega} j'_0 dx + \int_{\Gamma} j_0(\cdot, u_\Omega) P_\Theta(\nu) d\sigma - \int_{\Omega} j'_0 dx \\ &= \int_{\Gamma} j_0(\cdot, u_\Omega) P_\Theta(\nu) d\sigma. \end{aligned} \quad (19)$$

On the other hand, by a similar argument, we have

$$\begin{aligned} j_1(\cdot, \nabla u_{\Omega_\epsilon}) - j_1(\cdot, \nabla u_\Omega) &= \|\nabla u_{\Omega_\epsilon} - \nabla \varphi_1\|^2 - \|\nabla u_\Omega - \nabla \varphi_1\|^2 \\ &= \|\nabla \delta u\|^2 + 2\langle \nabla u_\Omega - \nabla \varphi_1, \nabla \delta u \rangle \end{aligned} \quad (20)$$

Setting $\Psi_\Omega = \nabla(u_\Omega - \varphi_1)$ and $M_\epsilon = \left\| \frac{j_1(\cdot, \nabla u_{\Omega_\epsilon}) - j_1(\cdot, \nabla u_\Omega)}{\epsilon} - 2\langle \nabla(u_\Omega - \varphi_1), \nabla u' \rangle \right\|_{L^1(D)}$, we have

$$\begin{aligned} M_\epsilon &\leq \frac{1}{\epsilon} \|\nabla \delta u\|_{L^2(D)}^2 + 2 \int_D \left| \left\langle \Psi_\Omega, \nabla \left(\frac{\delta u}{\epsilon} - u' \right) \right\rangle \right| dx \\ &\leq \frac{1}{\epsilon} \|\nabla \delta u\|_{L^2(D)}^2 + 2 \|\Psi_\Omega\|_{L^2(D)}^2 \left\| \nabla \left(\frac{\delta u}{\epsilon} - u' \right) \right\|_{L^2(D)}^2. \end{aligned}$$

and, by using once more (18), we have

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \|\nabla \delta u\|_{L^2(D)}^2 = 0 \quad \text{and} \quad \lim_{\epsilon \rightarrow 0^+} 2 \|\Psi_\Omega\|_{L^2(D)}^2 \left\| \frac{\nabla \delta u}{\epsilon} - \nabla u' \right\|_{L^2(D)}^2 = 0.$$

Hence, we get

$$\lim_{\epsilon \rightarrow 0^+} M_\epsilon = 0,$$

that is, the function $\epsilon \mapsto j_1(\cdot, \nabla u_{\Omega_\epsilon}) \in L^1(D)$ is differentiable at 0^+ .

Setting $j'_1 = 2\langle \nabla(u_\Omega - \varphi_1), \nabla u' \rangle$ and applying Proposition 1 to the function $J_3(\epsilon)$, we obtain

$$\begin{aligned} \frac{d}{d\epsilon} (J_3(\epsilon)) \Big|_{\epsilon=0^+} &= \int_{\Omega} \frac{d}{d\epsilon} j_1(\cdot, \nabla u_{\Omega_\epsilon}) \Big|_{\epsilon=0^+} dx + \int_{\Gamma} j_1(\cdot, \nabla u_\Omega) P_\Theta(\nu) d\sigma - \int_{\Omega} \frac{d}{d\epsilon} j_1(\cdot, \nabla u_{\Omega_\epsilon}) \Big|_{\epsilon=0^+} dx \\ &= \int_{\Omega} j'_1 dx + \int_{\Gamma} j_1(\cdot, \nabla u_\Omega) P_\Theta(\nu) d\sigma - \int_{\Omega} j'_1 dx \\ &= \int_{\Gamma} j_1(\cdot, \nabla u_\Omega) P_\Theta(\nu) d\sigma. \end{aligned}$$

Therefore, we have obtained

$$\begin{aligned} \frac{d}{d\epsilon} (J_1(\epsilon)) \Big|_{\epsilon=0^+} &= \int_{\Gamma} j_0(\cdot, u_\Omega) P_\Theta(\nu) d\sigma, \\ \frac{d}{d\epsilon} (J_3(\epsilon)) \Big|_{\epsilon=0^+} &= \int_{\Gamma} j_1(\cdot, \nabla u_\Omega) P_\Theta(\nu) d\sigma, \end{aligned}$$

and, of course,

$$\alpha J_1(\epsilon) + \beta J_3(\epsilon) = \epsilon \alpha \int_{\Gamma} j_0(\cdot, u_\Omega) P_\Theta(\nu) d\sigma + \epsilon \beta \int_{\Gamma} j_1(\cdot, \nabla u_\Omega) P_\Theta(\nu) d\sigma + o(\epsilon). \quad (21)$$

Thus, it remains to compute the shape derivative of J_2 and J_4 . Using the formulas (20) and (17), we can write

$$\alpha J_2(\epsilon) + \beta J_4(\epsilon) = J_{2,4}(\epsilon) + Err(\epsilon).$$

where

$$J_{2,4}(\epsilon) = \alpha \int_{\Omega} 2(u_{\Omega} - \varphi_0)\delta u dx + \beta \int_{\Omega} 2\langle \nabla u_{\Omega} - \nabla \varphi_1, \nabla \delta u \rangle dx$$

and

$$Err(\epsilon) = \alpha \int_{\Omega} |\delta u|^2 dx + \beta \int_{\Omega} \|\nabla \delta u\|^2 dx.$$

First, by using the convergence in (18), we have

$$Err(\epsilon) = \alpha o_1(\epsilon) + \beta o_2(\epsilon) = o(\epsilon).$$

On the other hand, using Green's formulas, we have

$$\int_{\Omega} \langle \nabla u_{\Omega} - \nabla \varphi_1, \nabla \delta u \rangle dx = \int_{\Omega} -\Delta(u_{\Omega} - \varphi_1)\delta u dx + \int_{\partial\Omega} \partial_{\nu}(u_{\Omega} - \varphi_1)\delta u d\sigma.$$

Hence,

$$\begin{aligned} J_{2,4}(\epsilon) &= \alpha \int_{\Omega} 2(u_{\Omega} - \varphi_0)\delta u dx + 2\beta \int_{\Omega} -\Delta(u_{\Omega} - \varphi_1)\delta u dx + 2\beta \int_{\partial\Omega} 2\partial_{\nu}(u_{\Omega} - \varphi_1)\delta u d\sigma \\ &= \int_{\Omega} (2\alpha(u_{\Omega} - \varphi_0) - 2\beta\Delta(u_{\Omega} - \varphi_1))\delta u dx + 2\beta \int_{\partial\Omega} \partial_{\nu}(u_{\Omega} - \varphi_1)\delta u d\sigma. \end{aligned}$$

Now, let us introduce the unique solution ψ_0 of the following problem, called adjoint state problem,

$$\begin{cases} -\Delta\psi_0 = 2\alpha(u_{\Omega} - \varphi_0) - 2\beta\Delta(u_{\Omega} - \varphi_1) & \text{in } \Omega, \\ \psi_0 = 0 & \text{on } \Gamma = \partial\Omega. \end{cases} \quad (22)$$

Thus, by using Green's formulas, we can write

$$\begin{aligned} J_{2,4}(\epsilon) &= \int_{\Omega} -\Delta\psi_0\delta u dx + 2\beta \int_{\partial\Omega} \partial_{\nu}(u_{\Omega} - \varphi_1)\delta u d\sigma \\ &= \int_{\Omega} \langle \nabla\psi_0, \nabla\delta u \rangle dx - \int_{\partial\Omega} \partial_{\nu}\psi_0\delta u d\sigma + 2\beta \int_{\partial\Omega} \partial_{\nu}(u_{\Omega} - \varphi_1)\delta u d\sigma \\ &= \int_{\Omega} -\Delta\delta u\psi_0 dx + \int_{\partial\Omega} \partial_{\nu}\delta u\psi_0 d\sigma - \int_{\partial\Omega} \partial_{\nu}\psi_0\delta u d\sigma + 2\beta \int_{\partial\Omega} \partial_{\nu}(u_{\Omega} - \varphi_1)\delta u d\sigma. \end{aligned}$$

Using the fact that for all $\epsilon \in [0, 1[$, $\Omega \subseteq \Omega_{\epsilon}$ and $u_{\Omega_{\epsilon}}$ and u_{Ω} are respectively solutions of problems (16) and (15), we get that

$$-\Delta\delta u = 0 \text{ in } \Omega. \quad (23)$$

On the other hand, we know that $\psi_0 \in H_0^1(\Omega)$. Hence,

$$J_{2,4}(\epsilon) = - \int_{\partial\Omega} \partial_{\nu}\psi_0\delta u d\sigma + 2\beta \int_{\partial\Omega} \partial_{\nu}(u_{\Omega} - \varphi_1)\delta u d\sigma.$$

Let us set

$$\Xi_1(\epsilon) = - \int_{\partial\Omega} \partial_{\nu}\psi_0\delta u d\sigma \quad \text{and} \quad \Xi_2(\epsilon) = 2\beta \int_{\partial\Omega} \partial_{\nu}(u_{\Omega} - \varphi_1)\delta u d\sigma.$$

In order to deal with the computation of the shape derivatives of the functions Ξ_1 and Ξ_2 , let us introduce the solution ψ_ϵ of the following problem:

$$\begin{cases} -\Delta\psi_\epsilon = 2\alpha(u_\Omega - \varphi_0) - 2\beta\Delta(u_\Omega - \varphi_1) & \text{in } \Omega_\epsilon, \\ \partial_\nu\psi_\epsilon = 0 & \text{on } \Gamma_\epsilon = \partial\Omega_\epsilon. \end{cases} \quad (24)$$

Thus, we have

$$\int_{\partial\Omega_\epsilon} \partial_\nu\psi_\epsilon \delta u d\sigma = 0.$$

Hence, $\Xi_1(\epsilon)$ can be written

$$\Xi_1(\epsilon) = - \int_{\partial\Omega} \partial_\nu\psi_0 \delta u d\sigma + \int_{\partial\Omega_\epsilon} \partial_\nu\psi_\epsilon \delta u d\sigma.$$

On the other hand, we know that u_ϵ is the solution of (16), so we have

$$\int_{\partial\Omega_\epsilon} \partial_\nu(u_{\Omega_\epsilon} - \varphi_1) u_{\Omega_\epsilon} d\sigma = 0.$$

Thus, Ξ_2 can be written

$$\Xi_2(\epsilon) = 2\beta \int_{\partial\Omega} \partial_\nu(u_\Omega - \varphi_1) \delta u d\sigma - 2\beta \int_{\partial\Omega_\epsilon} \partial_\nu(u_{\Omega_\epsilon} - \varphi_1) u_{\Omega_\epsilon} d\sigma.$$

Let us calculate the derivative of Ξ_1 . By using Green's formulas, we have

$$\begin{aligned} \Xi_1(\epsilon) &= - \int_{\partial\Omega} \partial_\nu\psi_0 \delta u d\sigma + \int_{\partial\Omega_\epsilon} \partial_\nu\psi_\epsilon \delta u d\sigma \\ &= - \int_{\Omega} \Delta\psi_0 \delta u dx - \int_{\Omega} \langle \nabla\psi_0, \nabla\delta u \rangle dx + \int_{\Omega_\epsilon} \Delta\psi_\epsilon \delta u dx + \int_{\Omega_\epsilon} \langle \nabla\psi_\epsilon, \nabla\delta u \rangle dx. \end{aligned}$$

Using the fact that $\Omega \subseteq \Omega_\epsilon$, ψ_0 and ψ_{Ω_ϵ} are respectively solutions of problems (22) and (24) and the fact that $-\Delta u_\Omega = f$ in Ω , we obtain

$$\begin{aligned} \Xi_1(\epsilon) &= \int_{\Omega} (2\alpha(u_\Omega - \varphi_0) - 2\beta\Delta(u_\Omega - \varphi_1)) \delta u dx - \int_{\Omega} \langle \nabla\psi_0, \nabla\delta u \rangle dx \\ &\quad - \int_{\Omega_\epsilon} (2\alpha(u_\Omega - \varphi_0) - 2\beta\Delta(u_\Omega - \varphi_1)) \delta u dx + \int_{\Omega_\epsilon} \langle \nabla\psi_\epsilon, \nabla\delta u \rangle dx \\ &= \int_{\Omega} (2\alpha(u_\Omega - \varphi_0) + 2\beta(f + \Delta\varphi_1)) \delta u dx - \int_{\Omega} \langle \nabla\psi_0, \nabla\delta u \rangle dx \\ &\quad - \int_{\Omega_\epsilon} (2\alpha(u_\Omega - \varphi_0) + 2\beta(f + \Delta\varphi_1)) \delta u dx + \int_{\Omega_\epsilon} \langle \nabla\psi_\epsilon, \nabla\delta u \rangle dx \end{aligned}$$

Hence, we can write $\Xi_1(\epsilon) = \Upsilon_1(\epsilon) + \Upsilon_2(\epsilon)$, where

$$\Upsilon_1(\epsilon) = \int_{\Omega} (2\alpha(u_\Omega - \varphi_0) + 2\beta(f + \Delta\varphi_1)) \delta u dx - \int_{\Omega} \langle \nabla\psi_0, \nabla\delta u \rangle dx$$

and $\Upsilon_2(\epsilon) = - \int_{\Omega_\epsilon} (2\alpha(u_\Omega - \varphi_0) + 2\beta(f + \Delta\varphi_1)) \delta u dx + \int_{\Omega_\epsilon} \langle \nabla\psi_\epsilon, \nabla\delta u \rangle dx.$

Applying Proposition 2, the shape derivative of the function Υ_1 is given by

$$\frac{d}{d\epsilon} \Upsilon_1(\epsilon) \Big|_{\epsilon=0^+} = \int_{\Omega} (2\alpha(u_\Omega - \varphi_0) + 2\beta(f + \Delta\varphi_1)) u' dx - \int_{\Omega} \langle \nabla\psi_0, \nabla u' \rangle dx.$$

Next, using (18) and Proposition 2, we get the shape derivative of the function Υ_2 :

$$\left. \frac{d}{d\epsilon} \Upsilon_2(\epsilon) \right|_{\epsilon=0^+} = - \int_{\Omega} (2\alpha(u_{\Omega} - \varphi_0) + 2\beta(f + \Delta\varphi_1))u' dx + \int_{\Omega} \langle \nabla\psi_0, \nabla u' \rangle dx.$$

Hence, the shape derivative of $\Xi_1(\epsilon)$ is given by $\left. \frac{d}{d\epsilon} \Xi_1(\epsilon) \right|_{\epsilon=0^+} = 0$.

Let us now calculate the derivative of Ξ_2 . By using Green's formulas, we have

$$\begin{aligned} \Xi_2(\epsilon) &= 2\beta \int_{\partial\Omega} \partial_{\nu}(u_{\Omega} - \varphi_1) \delta u d\sigma - 2\beta \int_{\partial\Omega_{\epsilon}} \partial_{\nu}(u_{\Omega_{\epsilon}} - \varphi_1) u_{\Omega_{\epsilon}} d\sigma \\ &= 2\beta \int_{\partial\Omega} \partial_{\nu}(u_{\Omega} - \varphi_1) u_{\Omega_{\epsilon}} d\sigma - 2\beta \int_{\partial\Omega_{\epsilon}} \partial_{\nu}(u_{\Omega_{\epsilon}} - \varphi_1) u_{\Omega_{\epsilon}} d\sigma + 2\beta \int_{\partial\Omega} \partial_{\nu}(u_{\Omega} - \varphi_1) u_{\Omega} d\sigma \\ &= 2\beta \int_{\Omega} \Delta(u_{\Omega} - \varphi_1) u_{\Omega_{\epsilon}} dx + 2\beta \int_{\Omega} \langle \nabla(u_{\Omega} - \varphi_1), \nabla u_{\Omega_{\epsilon}} \rangle dx \\ &\quad - 2\beta \int_{\Omega_{\epsilon}} \Delta(u_{\Omega_{\epsilon}} - \varphi_1) u_{\Omega_{\epsilon}} dx - 2\beta \int_{\Omega_{\epsilon}} \langle \nabla(u_{\Omega_{\epsilon}} - \varphi_1), \nabla u_{\Omega_{\epsilon}} \rangle dx + 2\beta \int_{\partial\Omega} \partial_{\nu}(u_{\Omega} - \varphi_1) u_{\Omega} d\sigma. \end{aligned}$$

By using the fact that u_{Ω} and $u_{\Omega_{\epsilon}}$ are respectively the solution of the problems (15) (with $g = 0$) and (16), we get

$$\begin{aligned} \Xi_2(\epsilon) &= -2\beta \int_{\Omega} (f + \Delta\varphi_1) u_{\Omega_{\epsilon}} dx + 2\beta \int_{\Omega} \langle \nabla(u_{\Omega} - \varphi_1), \nabla u_{\Omega_{\epsilon}} \rangle dx \\ &\quad + 2\beta \int_{\Omega_{\epsilon}} (f + \Delta\varphi_1) u_{\Omega_{\epsilon}} dx - 2\beta \int_{\Omega_{\epsilon}} \langle \nabla(u_{\Omega_{\epsilon}} - \varphi_1), \nabla u_{\Omega_{\epsilon}} \rangle dx. \end{aligned}$$

So, we can write

$$\Xi_2(\epsilon) = \Sigma_1(\epsilon) + \Sigma_2(\epsilon),$$

where

$$\Sigma_1(\epsilon) = -2\beta \int_{\Omega} (f + \Delta\varphi_1) u_{\Omega_{\epsilon}} dx + 2\beta \int_{\Omega} \langle \nabla(u_{\Omega} - \varphi_1), \nabla u_{\Omega_{\epsilon}} \rangle dx$$

and

$$\Sigma_2(\epsilon) = 2\beta \int_{\Omega_{\epsilon}} (f + \Delta\varphi_1) u_{\Omega_{\epsilon}} dx - 2\beta \int_{\Omega_{\epsilon}} \langle \nabla(u_{\Omega_{\epsilon}} - \varphi_1), \nabla u_{\Omega_{\epsilon}} \rangle dx.$$

Now, applying (18) and Proposition 2, we have

$$\left. \frac{d}{d\epsilon} \Sigma_1(\epsilon) \right|_{\epsilon=0^+} = -2\beta \int_{\Omega} (f + \Delta\varphi_1) u' dx + 2\beta \int_{\Omega} \langle \nabla(u_{\Omega} - \varphi_1), \nabla u' \rangle dx$$

and

$$\begin{aligned} \left. \frac{d}{d\epsilon} \Sigma_2(\epsilon) \right|_{\epsilon=0^+} &= 2\beta \int_{\Omega} (f + \Delta\varphi_1) u' dx - 2\beta \int_{\Omega} \langle \nabla u', \nabla u_{\Omega} \rangle dx + 2\beta \int_{\Omega} \langle \nabla(u_{\Omega} - \varphi_1), \nabla u' \rangle dx \\ &= 2\beta \int_{\Gamma} (f + \Delta\varphi_1) u_{\Omega} P_{\Theta}(\nu) d\sigma - 2\beta \int_{\Gamma} \langle \nabla(u_{\Omega} - \varphi_1), \nabla u_{\Omega} \rangle P_{\Theta}(\nu) d\sigma \\ &= -2\beta \int_{\Gamma} \langle \nabla(u_{\Omega} - \varphi_1), \nabla u_{\Omega} \rangle P_{\Theta}(\nu) d\sigma. \end{aligned}$$

Hence,

$$\left. \frac{d}{d\epsilon} \Xi_2(\epsilon) \right|_{\epsilon=0^+} = 2\beta \int_{\Omega} \langle \nabla u', \nabla u_{\Omega} \rangle dx - 2\beta \int_{\Gamma} \langle \nabla(u_{\Omega} - \varphi_1), \nabla u_{\Omega} \rangle P_{\Theta}(\nu) d\sigma,$$

since $u_\Omega = g = 0$ on $\Gamma = \partial\Omega$. Furthermore, we have $\int_\Omega \langle \nabla u', \nabla u_\Omega \rangle dx = 0$. Indeed, note first that

$$\int_\Omega \langle \nabla u', \nabla u_\Omega \rangle dx = \frac{d}{d\epsilon} \int_\Omega \langle \nabla u_{\Omega_\epsilon}, \nabla u_\Omega \rangle dx \Big|_{\epsilon=0^+}.$$

Next, since u_Ω and u_{Ω_ϵ} are respectively the solution of the problems (15) and (16), it follows from Green's formula that

$$\int_\Omega \langle \nabla u_{\Omega_\epsilon}, \nabla u_\Omega \rangle dx = - \int_\Omega \Delta u_{\Omega_\epsilon} u_\Omega dx + \int_{\partial\Omega} \partial_\nu u_{\Omega_\epsilon} u_\Omega dx = \int_\Omega -f u_\Omega dx,$$

which implies that the term $\int_\Omega \langle \nabla u_{\Omega_\epsilon}, \nabla u_\Omega \rangle dx$ does not depend on ϵ . So, its derivative with respect to ϵ at 0 is null. Thus, we have obtained

$$\frac{d}{d\epsilon} J_{2,4}(\epsilon) \Big|_{\epsilon=0^+} = \frac{d}{d\epsilon} \Xi_2(\epsilon) \Big|_{\epsilon=0^+} = -2\beta \int_\Gamma \langle \nabla(u_\Omega - \varphi_1), \nabla u_\Omega \rangle P_\Theta(\nu) d\sigma,$$

or, in another form,

$$J_{2,4}(\epsilon) = \alpha J_2(\epsilon) + \beta J_4(\epsilon) = \epsilon \left(-2\beta \int_\Gamma \langle \nabla(u_\Omega - \varphi_1), \nabla u_\Omega \rangle P_\Theta(\nu) d\sigma \right) + o(\epsilon). \quad (25)$$

Consequently, it follows from equation (21) and (25) and the fact that $J_i(0) = 0$, $i = 1, 2, 3, 4$, that

$$\Delta \mathcal{J} = \alpha(J_1(\epsilon) + J_2(\epsilon)) + \beta(J_3(\epsilon) + J_4(\epsilon)) = \epsilon \int_\Gamma \mathcal{H}(x) P_\Theta(\nu(x)) d\sigma(x) + o(\epsilon),$$

where

$$\mathcal{H} = \alpha j_0(\cdot, u_\Omega) + \beta j_1(\cdot, \nabla u_\Omega) - 2\beta \langle \nabla(u_\Omega - \varphi_1), \nabla u_\Omega \rangle.$$

Thus,

$$\delta \mathcal{J}(\Omega)[\Theta] = \int_\Gamma \mathcal{H}(x) P_\Theta(\nu(x)) d\sigma(x). \quad (26)$$

We are now in position to state the main result of this paper.

Theorem 3. *Suppose that the assumptions **(H)** are satisfied. Let $\Omega \in \mathcal{U}$, $\Theta \in \mathcal{O}$ and $\Omega_\epsilon = \Omega + \epsilon\Theta$, for $\epsilon \in [0, 1]$. Assume further that $0 \in \Theta$ and that Θ is strongly convex. Then, the shape derivative at Ω in the direction Θ of the constrained functional \mathcal{J} expressed by (14), is given by*

$$\delta \mathcal{J}(\Omega)[\Theta] = \int_\Gamma \mathcal{H}(x) P_\Theta(\nu(x)) d\sigma(x), \quad (27)$$

where

$$\mathcal{H} = \alpha j_0(\cdot, u_\Omega) + \beta j_1(\cdot, \nabla u_\Omega) - 2\beta \langle \nabla(u_\Omega - \varphi_1), \nabla u_\Omega \rangle.$$

5 Description of the numerical setting and outline of the algorithm

The proposed numerical optimization algorithm for solving the problem (3) is based on a gradient method.

5.1 Numerical algorithm

Based on the shape derivative formula (26) of the cost functional gradient of \mathcal{J} , the computation of the optimal shape is done using the gradient numerical method summarized in the following algorithm where we take $\alpha = 1 - t$ and $\beta = t$, $t \in [0, 1]$.

(1) Initialization.

- Choose an initial domain $\Omega_0 \in \mathcal{U}$;
- Fix step size $\rho \in]0, 1[$ and a precision Eps.

(2) Main part, for iteration $k=0, \dots$

- (i) Calculate the respective solution $u_k = u_{\Omega_k}$ of the state problem (15) on Ω_k .
- (ii) Calculate the respective solution $\psi_k = \psi_{\Omega_k}$ the adjoint state problem (22) Ω_k .
- (iii) ◦ Extract u_k , ψ_k , ∇u_k and $\nabla \psi_k$ on $\Gamma_k = \partial\Omega_k$.
 - Compute \mathcal{H}_k on Γ_k by the relation

$$\mathcal{H}_k = (1 - t)j_0(\cdot, u_k) + tj_1(\cdot, \nabla u_k) - 2t\langle \nabla(u_k - \varphi_1), \nabla u_k \rangle.$$

- (iv) Compute $P_k = P_{\Omega_k}$.
- (v) Compute \widehat{P}_k the solution of

$$\arg \min_{\varphi \in \mathcal{P}} \Lambda_k(\varphi) := \int_{\Gamma_k} \mathcal{H}_k(x) \varphi(\nu(x)) ds \quad (28)$$

where

$$\mathcal{P} = \{\Phi \in C(\mathbb{R}^n) / \Phi \text{ is convex and homogeneous of degree 1 and } P_{B(0,r)} \leq \Phi \leq P_D\}.$$

where $B(0, r)$ is the open ball of center 0 and radius r in \mathbb{R}^n .

- (vi) ◦ Compute

$$\Omega_{k+1} = \Omega_k + \rho \Theta_k.$$

where the domain Θ_k is given by

$$\Theta_k := \partial \widehat{P}_k(0) = \left\{ l \in \mathbb{R}^n / \widehat{P}_k(x) \geq \langle l, x \rangle, \forall x \in \mathbb{R}^n \right\}$$

(3) End criteria.

- if $\|\Lambda_k(\widehat{P}_k)\| \leq \text{Eps}$, **Return** Ω_k .
- else, **Back to previous step (2)**.

Remark 1. • Note that, the shape derivative of a fairly general class of shape functionals $J(\Omega)$ in direction of a vector field ϑ has the generic form:

$$J'(\Omega)(\vartheta) = \int_{\partial\Omega} g(\vartheta(x), \nu(x)) d\sigma(x) =: \langle g|_{\Gamma}, \langle \vartheta(x), \nu(x) \rangle \rangle_{L^2(\partial\Omega)}. \quad (29)$$

where the scalar function $g : \partial\Omega \rightarrow \mathbb{R}$ is the shape gradient of J with respect to the $L^2(\partial\Omega)$ inner product. This statement is referred to as the Hadamard structure theorem for shape derivatives (Sokolowski & Zolesio, 1992). In the particular case of convexity constraint in the family of admissible domains, according to Theorem 1, this structure theorem becomes

$$\delta J(\Omega)[\Theta] := \int_{\partial\Omega} f_{\partial\Omega}(x) P_{\Theta}(\nu(x)) d\sigma(x) = \langle f_{\partial\Omega}, P_{\Theta}(\nu) \rangle_{L^2(\partial\Omega)}. \quad (30)$$

Here, $\delta J(\Omega)[\Theta]$ depends only on the normal component of P_{Θ} on the boundary $\partial\Omega$.

- The expression (30) allows us to easily deduce the direction of descent, as it was summarized in the above algorithm, because the sequence of domains $(\Omega_k)_{k \in \mathbb{N}}$ is constructed in such a way that $(\mathcal{J}(\Omega_k))_{k \in \mathbb{N}}$ is decreasing. Indeed, let $k \in \mathbb{N}^*$, then, for a small $\rho \in]0, 1[$, we have

$$\begin{aligned} \mathcal{J}(\Omega_{k+1}) - \mathcal{J}(\Omega_k) &= \mathcal{J}(\Omega_k + \rho\Theta_k) - \mathcal{J}(\Omega_k) \\ &= \rho \left(\int_{\partial\Omega_k} \mathcal{H}_k P_{\Theta_k} \circ \nu_k d\sigma \right) + o(\rho). \end{aligned}$$

Now, if we denote by $\Lambda_k(p) = \int_{\partial\Omega_k} \mathcal{H}_k p \circ \nu_k d\sigma$, since $\widehat{P}_k = P_{\Theta_k}$ is a solution of $\arg \min_{p \in \mathcal{E}} \Lambda_k(p)$, then

$$\Lambda_k(\widehat{P}_k) = \int_{\partial\Omega_k} \mathcal{H}_k P_{\Theta_k} \circ \nu_k d\sigma \leq \Lambda_k(0) = 0,$$

which guarantees the decrease of the objective function \mathcal{J} . Consequently, $\widehat{\Omega}_k$ defines a descent direction for \mathcal{J} .

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